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RESEARCH NOTE

Moments structure of ℓ_1-stochastic volatility models

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Abstract We consider Taylor's stochastic volatility model (SVM) when the innovations of the hidden log-volatility process have a Laplace distribution (ℓ_1 exponential density), rather than the standard Gaussian distribution (ℓ_2) usually employed. Recently many investigations have employed ℓ_1 metric to allow better modeling of the abrupt changes of regime observed in financial time series. However, the estimation of SVM is known to be difficult because it is a non-linear with an hidden markov process. Moreover, an additional difficulty yielded by the use of ℓ_1 metric is the not differentiability of the likelihood function. An alternative consists in using a generalized or efficient method-of-moments (GMM/EMM) estimation. For this purpose, we derive here the moments and autocovariance function of such ℓ_1-based stochastic volatility models.

Keywords Stochastic volatility model · Laplace innovations · Autocovariance function · Variance gamma model

JEL Classification C22

1 Introduction

Stochastic volatility (SV) model introduced by Taylor (1986) is the empirical discrete-time version of continuous-time models of finance theory, in particular of the models used in option pricing problem (Hull and White 1987). It can also be regarded as an Euler discretisation of a diffusion. SV models are a good alternative to ARCH models, with theoretical properties that fit in a more appropriate way the stylized features observed in finance. In particular, the

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log-normal SV model seems to capture more leptokurticity of the marginal distributions of the financial data than the conditional GARCH model (Shephard 1996). Taylor's SV model can be defined as

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t && (1a)\\
\sigma_t &= \exp(h_t) && (1b)\\
h_t &= \mu + \phi(h_{t-1} - \mu) + \eta_t && (1c),
\end{aligned}
\tag{1}
$$

for $t \leq T$, where y_t are the log-returns over a unit time period, σ_t are the volatilities, $\phi > 0$ and μ are the two parameters respectively representing the autoregressive coefficient and the drift of the log-volatilities h_t, ϵ_t and η_t are two mutually and serially independent innovations centered around zero with respective variance one and σ_η. In general, η_t and ϵ_t are assumed to be Gaussian. The properties of model (1) are given by Taylor (1994) and Shephard (1996). In particular, Andersen (1994) shows the condition $\log(\phi) < 0$ guarantees the strictly stationarity and the ergodicity of the volatility process (Andersen 1994, Theorem 2.1, p. 82).

A stylized feature of financial time series is volatility clustering and abrupt changes of volatility regimes. In the recent years, modeling abrupt changes in the volatility has been one of the main challenge in the financial literature because they provide a good explanation of the (global) non-stationary feature of the volatility (See Starica and Granger 2005, among others). Abrupt changes can be obtained by SV models if the stochastic innovations in (1c) can take large (either positive or negative) values with a greater probability than with Gaussian innovations. To that aim we propose to employ Laplace(θ, λ) innovations with density

$$
f(\eta_t; \beta, \lambda) = \frac{\lambda}{2} \exp(-\lambda |\eta_t - \theta|) \tag{2}
$$

centered around $\theta = 0$. The use of ℓ_1-based innovations has been successful to recover process with abrupt changes with wavelet-based estimators (Donoho and Johnstone 1994) and Markov random field-based estimators (Sardy and Tseng 2004). This idea bears similarity with the investigation of Madan and Seneta (1990), Madan et al. (1998) which considers a Variance Gamma (VG) model for modeling the non-Gaussian nature of stock market returns and future Index prices.[1] The VG model can be expressed by Eq. 1a, where σ_t follows a Gamma process and the continuous-time stochastic process which is consistent with this VG model is given by

$$
S(t) = S(0) \exp(L(t)),
$$

where $S(t)$ are the stock prices and $L(t)$ is a Laplace motion (Kotz et al. 2001). Hence, the Laplace motion is defined as a Brownian motion, denoted $B(\tau)$, evaluated at random time distributed as a Gamma process, denoted $\gamma(t)$. This Brownian is said subordinated to the process $\gamma(t)$. Therefore, the Laplace motion is defined as $L(t) \overset{d}{=} B(\gamma(t))$. Another appeal of the Laplace motion is it can be written as a compound Poisson process with independent and random jumps. In this sense it is a pure jumps process able then to capture abrupt changes. The Laplace SV model we study here also extends from VG model in that the additional parameter ϕ allows control of the level of persistence of the time series.

However, two pitfalls are combined in the estimation of such model: in addition to the issue of the hidden Markov process, the likelihood function of this model is not differentiable due to the use of the ℓ_1 exponential density, which makes the likelihood-based estimation methods trickier to apply here than in the Gaussian version of (1) (Chib et al. 2002).

[1] Indeed, the estimates of the risk neutral processes are known to not by Gaussian.

We explore the moment properties of such models. Our results could be exploited for instance in a method of moment estimation procedure (Taylor 1986; Melino and Turnbull 1990; Andersen and Sørensen 1996; Andersen et al. 1999).

2 Moment properties

Andersen (1994) shows that for $\phi > 0$, the rth moment of h_t exists if and only if $\phi^r < 1$ and $\mathbb{E}\left(\eta_t^r\right) < \infty$. Therefore, for (1c) with η_t iid Laplace(0, λ), first-order and second-order (unconditional) moments of h_t exist and are given by $\mathbb{E}\left(h_t\right) = \mu$ and $\mathbb{V}\left(h_t\right) = \frac{2}{\lambda^2(1-\phi^2)}$. Moreover, the probability distribution function of the marginal distribution of $(h_t - \mu)$ is given by the following proposition.

Proposition 1 *The marginal probability distribution function of the process* $\{h_t - \mu\}_t$ *in the* ℓ_1*-SVM defined by* (1) − (2) *is given by*

$$f\left(h_t - \mu\right) = \frac{\lambda}{2}\sum_{i=0}^{\infty} |\phi|^{-i}\, g_i\left(\phi\right) \exp\left(-\lambda\left|h_t - \mu\right| \phi^{-i}\right), \tag{3}$$

where the function $g_i\left(\phi\right)$ *is:*

$$g_i\left(\phi\right) = (-1)^i \prod_{k=1}^{i}\left(\frac{\phi^{2k}}{1-\phi^{2k}}\right)\left(\prod_{k=1}^{\infty}\left(1-\phi^{2k}\right)\right)^{-1}, \quad for\, i \in \mathbb{N},$$

with the convention that $\prod_{i=1}^{0}\left(\cdot\right) \equiv 1.$

The moments of $\{h_t - \mu\}_t$ *are given by* $\mathbb{E}\left((h_t - \mu)^r\right) = \frac{r!}{\lambda^r}\sum_{i=0}^{\infty}\phi^{ir} g_i\left(\phi\right)$, *for r even, and zero otherwise.*

Proof Let $\{X_t\}_t$ be a stationary process generated by an ARMA(p,q) process such that $\Phi\left(L\right) X_t = \Theta\left(L\right) \eta_t$, where η_t is independent, identically Laplace (double-exponential) distributed, $\Phi\left(L\right) = 1 - \sum_{j=1}^{p}\phi_j L^j$, $\Theta\left(L\right) = 1 - \sum_{j=1}^{q}\theta_j L^j$ with all roots of $\Phi\left(L\right) = 0$ and $\Theta(L) = 0$ lying outside the unit circle. The MA(∞) representation is given by $X_t = \Psi\left(L\right) \eta_t$, where $\Psi(L) = \Phi^{-1}(L)\,\Theta(L) = \sum_{j=0}^{\infty}\psi_j L^j$. Damsleth and El-Shaarawi (1989, pp. 62–63) give the marginal probability distribution function of X_t. This pdf is given by:

$$f\left(x\right) = \frac{\lambda}{2}\sum_{j=0}^{\infty}|\psi_j|^{-1} g_j(\psi_j) \exp(-\lambda|x/\psi_j|),$$

where

$$g_j(\psi_j) = \prod_{i=0, i\neq j}^{\infty}(1 - |\psi_i/\psi_j|^2)^{-1}.$$

In the ℓ_1- SVM, $\{h_t - \mu\}_t$ is generated by an AR(1), therefore its marginal pdf is obtained using $\psi_j = \phi^j$, where $\phi \in (0, 1)$. □

Since the marginal distribution of h_t is symmetric about zero, the odd unconditional (centered) moments of y_t are zeros. The following theorem gives the unconditional moments function of y_t.

Theorem 1 *For the ℓ_1-SVM defined by* $(1)-(2)$ *which satisfies* $\phi \in (0,1)$, *the* rth *unconditional moment of* y_t *exists if* $\lambda > r$ *and is given by:*

$$m_R = \frac{r!\lambda^2 \exp(r\mu)}{2^{\frac{r}{2}}\,(r/2)!}\sum_{i=0}^{\infty}\frac{\phi^{-2i} g_i(\phi)}{\left(\lambda\phi^{-i}+r\right)\left(\lambda\phi^{-i}-r\right)},$$

for r *even and* $m_r = 0$ *otherwise.*

Proof Using h_t independent of ϵ_t, which is iid standard Gaussian, and denoting $\mathbb{E}\left(y_t^r\right) = m_r$, we have $m_r = \mathbb{E}\left(\exp(rh_t)\right)\mathbb{E}\left(\epsilon_t^r\right) = \mathbb{E}\left(\exp\left(r\left(h_t-\mu\right)\right)\right)\exp(r\mu)\,\mathbb{E}\left(\epsilon_t^r\right)$.

We use the following Corollary of the dominated convergence theorem to derive $\mathbb{E}\left(\exp\left(r\left(h_t-\mu\right)\right)\right)$.

Corollary *Let* $\{f_n\}_{n\in N}$ *be a sequence of integrable functions on a measure space* $(\Omega, F, \mathbb{P})$ *such that we have* $\sum_{n=1}^{\infty}\int_{\Omega}|f_n|\,\mathrm{d}\mathbb{P} < \infty$, *then* $\sum_{n=1}^{\infty} f_n(x)$ *converges almost everywhere to an integrable function* f *and* $\int_{\Omega}\sum_{n=1}^{\infty} f_n(x)\mathrm{d}\mathbb{P} = \sum_{n=1}^{\infty}\int_{\Omega} f_n(x)\,\mathrm{d}\mathbb{P}$.

Let us set $f_i(h) = (\lambda/2)\,\phi^{-i} g_i(\phi)\exp\left(rh - \lambda|h|\phi^{-i}\right)$, we have

$$\int_{\mathbb{R}} |f_i(h)|\,\mathrm{d}h = -\frac{|g_i(\phi)|\lambda^2}{r^2\phi^{2i}-\lambda^2},$$

for $\lambda > r$ and $\phi \in (0,1)$. Denoting $G_i = -\frac{g_i(\phi)\lambda^2}{r^2\phi^{2i}-\lambda^2}$, we verify that $\sum_{i=1}^{\infty} G_i < \infty$ for $|\phi| < 1$ through

$$\lim_{i\to\infty}|G_{i+1}/G_i| = \lim_{i\to\infty}\frac{\phi^{2(i+1)}\left(r^2\phi^{2i}-\lambda^2\right)}{\left(1-\phi^{2(i+1)}\right)\left(r^2\phi^{2(i+1)}-\lambda^2\right)} = 0.$$

Here $\lim_{i\to\infty}|G_{i+1}/G_i| < 1$, so $\sum_{i=1}^{\infty}|G_i|$ and $\sum_{i=1}^{\infty} G_i$ converge.

Therefore, we can write:

$$\begin{aligned}\mathbb{E}\left(\exp\left(r\left(h_t-\mu\right)\right)\right) &= \int_{\mathbb{R}}\exp(rh)\,(\lambda/2)\sum_{i=0}^{\infty}\phi^{-i} g_i(\phi)\exp\left(-\lambda\,|h|\,\phi^{-i}\right)\mathrm{d}h\\ &= (\lambda/2)\sum_{i=0}^{\infty}\phi^{-i} g_i(\phi)\int_{\mathbb{R}}\exp\left(rh-\lambda\,|h|\,\phi^{-i}\right)\mathrm{d}h.\end{aligned}$$

Straightforward algebra leads to the moments function. □

In particular from Theorem 1 we obtain the kurtosis of the marginal distribution of y_t, defined as $\kappa_y = m_4/m_2^2$:

$$\kappa_y = 3\left(\lambda^2\sum_{i=0}^{\infty}\frac{g_i(\phi)}{\left(\lambda^2-4\phi^{2i}\right)}\right)^{-2}\sum_{i=0}^{\infty}\frac{g_i(\phi)}{\left(\lambda^2-16\phi^{2i}\right)},\quad \text{for } \lambda > 4.$$

Figure 1 compares the kurtosis functions between the Laplace and log-normal SV models. Hence, for ϕ fixed, the unconditional distribution of the process y_t gets more leptokurtic as λ decreases. Therefore, the parameter λ may be regarded as a measure of the long tailedness.

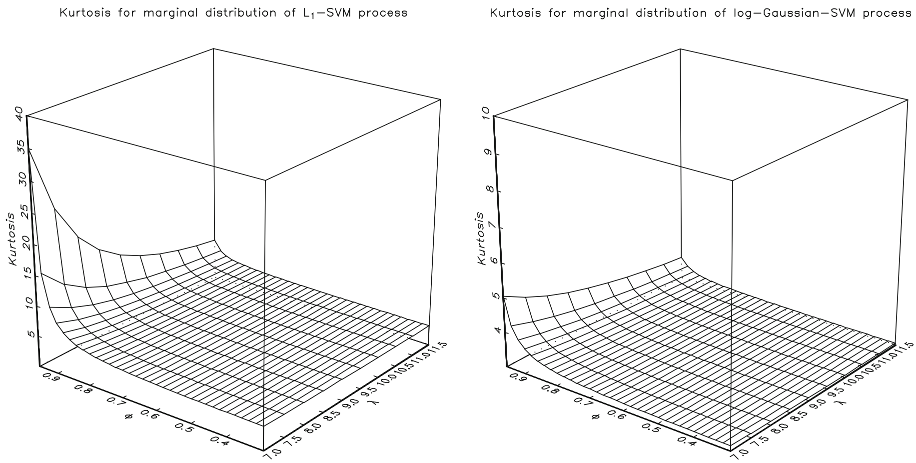

Fig. 1 Kurtosis functions when y_t follows: (*left*) a Laplace-SVM; (*right*) the corresponding log-normal SVM, where $\kappa_y = 3\exp\left(\frac{\sigma_\eta^2}{1-\phi^2}\right)$ and $\sigma_\eta^2 = 2/\lambda^2$

Theorem 2 *For* $\lambda > 4$, *the autocovariance function of the squared values of* y_t, *denoted* $\gamma_s = \operatorname{cov}\left(y_t^2 y_{t-s}^2\right)$, *for* $s \in \mathbb{N}^*$, *is finite and given by:*

$$\gamma_s = \exp(4\mu)\left(A_1(s,\phi,\lambda)\, A_2(s,\phi,\lambda) - \lambda^4 A_3(\phi,\lambda)^2\right),$$

where $A_1(s,\phi,\lambda) = \prod_{i=0}^{s-1} \frac{\lambda^2}{\lambda^2 - 4\phi^{2i}}$, $A_2(s,\phi,\lambda) = \prod_{i=s}^{\infty} \frac{\lambda^2}{\lambda^2 - 4(1+\phi^s)^2\phi^{2(i-s)}}$, *and* $A_3(\phi,\lambda) = \sum_{i=0}^{\infty} \frac{\phi^{-2i} g_i(\phi)}{\left(\lambda|\phi|^{-i}+2\right)\left(\lambda|\phi|^{-i}-2\right)}$, *with the convention* $\prod_{i=0}^{-1}(\cdot) \equiv 1$.

The autocorrelation function $\rho_s = \gamma_s / m_4$ *is given by:*

$$\rho_s = \frac{A_1(s,\phi,\lambda)\, A_2(s,\phi,\lambda) - \lambda^4 A_3(\phi,\lambda)^2}{3\lambda^2 A_0(\phi,\lambda) - \lambda^4 A_3(\phi,\lambda)^2},$$

with $A_0(\phi,\lambda) = \sum_{i=0}^{\infty} \frac{g_i(\phi)}{\left(\lambda^2 - 16\phi^{2i}\right)}$.

Proof Using h_t independent of ϵ_t, we write $\gamma_s = \mathbb{E}\left(y_t^2 y_{t-s}^2\right) - \mathbb{E}\left(y_t^2\right)^2 = \mathbb{E}(\exp(2h_t + 2h_{t-s})) - \mathbb{E}\left(y_t^2\right)^2$, where the second term is given by Theorem 1: $\mathbb{E}\left(y_t^2\right)^2 = \lambda^2 \exp(4\mu)\, A_3(\phi,\lambda)$ where $A_3(\phi,\lambda)$ is given in the statement of the Theorem.

For the first term $\mathbb{E}(\exp(2h_t + 2h_{t-s}))$, we note that $h_t - \mu = \sum_{i=0}^{\infty} \phi^i \eta_{t-i}$ and $\sum_{i=0}^{\infty} \phi^i \eta_{t-i} + \sum_{i=0}^{\infty} \phi^i \eta_{t-s-i} = \sum_{i=0}^{s-1} \phi^i \eta_{t-i} + (1+\phi^s) \sum_{i=0}^{\infty} \phi^i \eta_{t-s-i}$.

We get $\exp(2h_t + 2h_{t-s}) = \exp(4\mu) \prod_{i=0}^{s-1} \exp(2\phi^i \eta_{t-i}) \prod_{i=s}^{\infty} \exp(2(1+\phi^s)\phi^{i-s}\eta_{t-i})$.

Using $\mathbb{E}(\exp(c\eta_t)) = \frac{\lambda^2}{(\lambda+c)(\lambda-c)}$, with c a constant, and taking the expectation of the expression of $\exp(2h_t + 2h_{t-s})$ we have: $\mathbb{E}(\exp(2h_t + 2h_{t-s})) = \exp(4\mu) A_1(s,\phi,\lambda) A_2(s,\phi,\lambda)$ where the functions $A_1(s,\phi,\lambda)$ and $A_2(s,\phi,\lambda)$ are defined in the Theorem's statement.

The expression of the autocorrelation function is then immediate using:

$$\rho_s = \frac{A_1(s,\phi,\lambda)\, A_2(s,\phi,\lambda) - \lambda^4 A_3(\phi,\lambda)^2}{\mathbb{V}\left(y_t^2\right)}.$$

□

References

Andersen, T.G.: Stochastic autoregressive volatility: a framework for volatility modeling. Math. Financ. **42**, 75–102 (1994)
Andersen, T.G., Sørensen, B.E.: GMM estimation of a stochastic volatility model: a Monte Carlo study. J. Bus. Econ. Stat. **14**, 328–352 (1996)
Andersen, T.G., Chung, H., Sørensen, B.E.: Efficient method of moments estimation of a stochastic volatility model: a Monte Carlo study. J. Econom. **91**, 61–87 (1999)
Chib, S., Nardari, F., Shephard, N.: Markov chain Monte Carlo methods for generalized stochastic volatility models. J. Econom. **108**, 281–316 (2002)
Damsleth, E., El-Shaarawi, H.: ARMA model with double-exponentially distributed noise. J. R. Stat. Soc. B. **51**, 61–69 (1989)
Donoho, D.L., Johnstone, I.M.: Ideal spatial adaptation via wavelet shrinkage. Biometrika **95**, 425–455 (1994)
Hull, J., White, A.: The pricing of options on assets with stochastic volatilities. J. Financ. **42**, 281–300 (1987)
Kotz, S., Kozubowski, T.J., Podgórski, K.: The Laplace Distribution and Generalizations: A Revisit with Applications to Communications. Economics, Engineering, and Finance. Birkhäuser, Boston (2001)
Madan, D.B., Seneta, E.: The variance gamma (V.G.) model for share market returns. J. Bus. **63**, 511–524 (1990)
Madan, D.B., Carr, P.P., Chang, E.C.: The variance gamma process and option pricing. Eur. Financ. Rev. **2**, 79–105 (1998)
Melino, A., Turnbull, S.M.: Pricing foreign currency options with stochastic volatility. J. Econom. **45**, 239–265 (1990)
Sardy, S., Tseng, P.: On the statistical analysis of smoothing by maximizing dirty Markov random field posterior distributions. J. Am. Stat. Assoc. **99**, 191–204 (2004)
Shephard, N: Statistical aspects of ARCH and stochastic volatility. In: Cox, D.R., Hinkley, D.V., Barndorff-Nielsen, O.E. (eds.) Time Series Models in Econometrics, Finance and Other Fields, pp. 1–67. Chapman and Hall, London (1996)
Starica, C., Granger, C.: Nonstationarities in stock returns. Rev. Econ. Stat. **87**, 503–522 (2005)
Taylor, S.J.: Modelling Financial Time Series. Wiley, Chichester (1986)
Taylor, S.J.: Modelling stochastic volatility: a review and comparative study. Math. Financ. **4**, 183–204 (1994)